

Fractional exclusion statistics in general systems with interaction

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Abstract. I show that fractional exclusion statistics (FES) is manifested in general interacting systems and I calculate the exclusion statistics parameters. Most importantly, I show that the mutual exclusion statistics parameters—when the presence of particles in one Hilbert space influences the dimension of another Hilbert space—are proportional to the dimension of the Hilbert space on which they act. This result, although surprising and different from the usual way of understanding the FES, renders this statistics consistent and valid in the thermodynamic limit, in accordance with the conjecture introduced in J. Phys. A: Math. Theor. **40** F1013 (2007).

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1. Introduction

Fractional exclusion statistics (FES), introduced by Haldane in Ref. [1] and with the thermodynamic properties calculated mainly by Isakov [2] and Wu [3], has received very much attention since its discovery and has been applied to many models of interacting systems (see for example Refs. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]). Several authors have also discussed the microscopic reason for the manifestation of FES [21, 22, 17, 18, 14, 15, 16, 19, 23, 20].

Iguchi and Sutherland [23] showed that liquids of particles in three dimensions, interacting through long-range forces exhibit the nature of quantum liquids with FES, the characteristics of the FES being determined by the interaction.

Murthy and Shankar [17] analysed a system of fermions in the Colagero-Sutherland model. The system has a constant density of states (DOS) (along the single particle energy axis) and has a total energy of

$$E(\{n_i\}) = \sum_i \epsilon_i n_i + \frac{V}{2\sigma} N(N-1), \quad (1)$$

where n_i is the population of the single particle state of energy ϵ_i , $\sigma = (\epsilon_i - \epsilon_{i-1})^{-1}$ (for any $i > 0$) is the DOS, V is the mean-field interaction potential, and N is the total number of particles in the system. By redistributing in an uneven way the interaction energy between the particles of the system and associating to the level i the quasi-particle energy

$$\tilde{\epsilon}_i = \epsilon_i + V\sigma^{-1} \sum_{j=0}^{i-1} n_j, \quad (2)$$

Murthy and Shankar obtained a gas with FES of parameter $\alpha = 1 + V$.

A model which is similar to that of Murthy and Shankar [17] has been employed also in Refs. [15, 16, 19] to describe anyons on the lowest Landau level, coupled chiral particles on a circle, or interacting bosons in two-dimensions.

In Refs. [24, 25] I showed that the same model, with a slight generalization, can lead to a condensation, which is a first order phase transition.

In this paper I will extend the method of Murthy and Shankar to systems of general DOS and any interaction potential, V_{ij} (where i and j label the single particle states) and I will show that such systems lead to a more general manifestation of FES. While in the Murthy and Shankar model we have only *direct* exclusion statistics (i.e. the exclusion statistics is manifested only in the subspace where the particles are inserted) of constant parameter, α , here, in the general case, we shall have also *mutual* statistics (acting from one subspace into another); therefore we shall have more complex parameters, denoted as α_{ij} . I will calculate explicitly the parameters α_{ij} and I will prove that the mutual parameters (α_{ij} , with $i \neq j$) are proportional to the dimension of the Hilbert subspace on which they act, verifying in this way the conjecture put forward in Ref. [26].

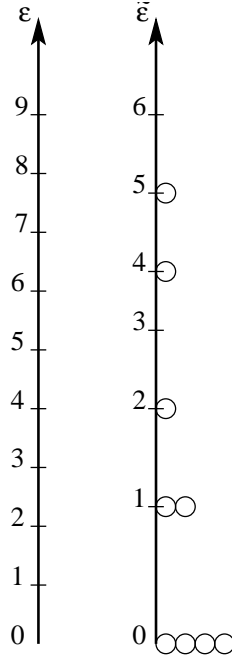


Figure 1. The single particle energy levels in the noninteracting system (left) and the corresponding quasiparticle energy levels (right, $\tilde{\epsilon}_i = \epsilon_i + \sum_{j=0}^{i-1} V_{ij}n_j$) when there are four particles on level 0, two on level 1, and one particle on each of the levels 2, 4, and 5. In this particular case I chose $\epsilon_i = i$ and $V_{ij} = 1/3$ for any i, j .

2. FES in systems with interaction

Let us generalize the model of Murthy and Shankar [17, 15, 16, 19] by writing the total energy as

$$E = \sum_i \epsilon_i n_i + \frac{1}{2} \sum_{ij} V_{ij} n_i n_j \quad (3)$$

and the quasiparticle energies as

$$\tilde{\epsilon}_i = \epsilon_i + \sum_{j=0}^{i-1} V_{ij} n_j + \frac{1}{2} V_{ii} n_i \quad (4)$$

(see figure 1). To make the calculations and the physical implications as clear as possible, we assume that we have bosons in the systems—in this way we shall not have to worry about adding a unit to the direct exclusion statistics parameters. I will also assume that the system is large enough, so that the spectrum is (quasi)continuous, of the (generally not constant) DOS, $\sigma(\epsilon)$. Then, assuming that V_{ij} depends only on the energies of the interacting particles, in Eq. (4) I drop the subscript i and I use ϵ as a variable, to write

$$\tilde{\epsilon} = \epsilon + \int_0^\epsilon V(\epsilon, \epsilon') \sigma(\epsilon') n(\epsilon') d\epsilon'. \quad (5)$$

In Eq. (5) I also ignored the term $V(\epsilon, \epsilon) n(\epsilon)$. Although this term, for $\epsilon = 0$, may cause a first order phase transition [24, 25], here I just want to emphasize the characteristics of the emerging FES and carrying along this term in the calculations would be useless. I

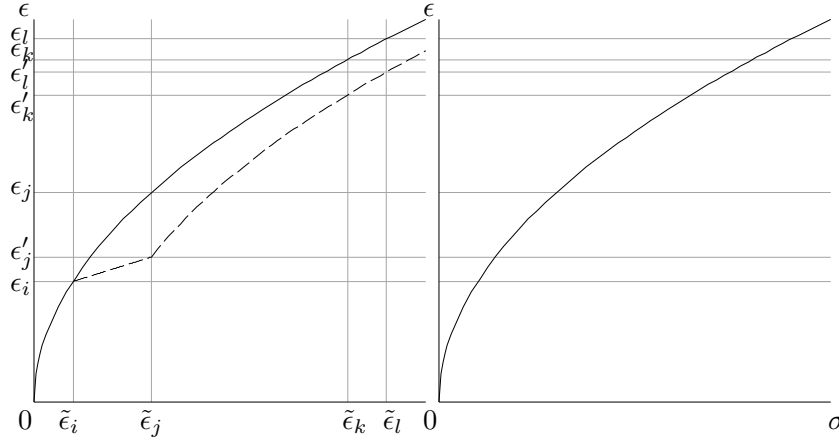


Figure 2. *Left:* $\epsilon(\tilde{\epsilon})$ before (solid curve) and after (dashed curve) the insertion of extra particles in the interval $(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$. This changes the values of ϵ_1 , ϵ_2 , and ϵ_3 into ϵ'_1 , ϵ'_2 , and ϵ'_3 , respectively. *To the right* I draw the inverse of $\sigma(\epsilon)$, to emphasize the change of the number of states ($\int \sigma(\epsilon) d\epsilon$) in each of the intervals after the insertion of particles. Both plots are schematic and are used only to illustrate the principle of calculation.

assume also that the function $\tilde{\epsilon}(\epsilon)$ is bijective, so that I can use freely its inverse, $\epsilon(\tilde{\epsilon})$. Since $\tilde{\epsilon}(\epsilon)$ and $\epsilon(\tilde{\epsilon})$ depend also on the populations of the energy levels below ϵ or below $\tilde{\epsilon}$, respectively, I shall use also the notations $\tilde{\epsilon}_{n(\epsilon' < \epsilon)}(\epsilon)$ and $\epsilon_{n(\tilde{\epsilon}' < \tilde{\epsilon})}(\tilde{\epsilon})$ whenever this will be needed for clarity.

If I denote the density of states along the $\tilde{\epsilon}$ axis by $\tilde{\sigma}(\tilde{\epsilon})$ and the number of particles between the energy levels $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$, by $N(\tilde{\epsilon}_1, \tilde{\epsilon}_2)$, then we have the relation

$$N(\tilde{\epsilon}_1, \tilde{\epsilon}_2) \equiv \int_{\tilde{\epsilon}_1}^{\tilde{\epsilon}_2} \tilde{\sigma}(\tilde{\epsilon}) n(\tilde{\epsilon}) d\tilde{\epsilon} = \int_{\epsilon(\tilde{\epsilon}_1)}^{\epsilon(\tilde{\epsilon}_2)} \sigma(\epsilon') n(\epsilon') d\epsilon',$$

where, obviously, $n(\tilde{\epsilon}) \equiv n[\epsilon(\tilde{\epsilon})]$.

To show the underlying FES character of the system, I use the coarse-graining of the energy axis $\tilde{\epsilon}$. I split the quasiparticle energy axis into intervals— $[\tilde{\epsilon}_0, \tilde{\epsilon}_1], \dots, [\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M], \dots$ —which are small, but still contain large enough numbers of particles and energy levels; the FES will be manifested between and within these intervals [21, 22, 17, 18, 14, 15, 16, 19, 23, 20]. To each $\tilde{\epsilon}_i$ it corresponds an $\epsilon_i \equiv \epsilon_{n(\epsilon < \epsilon_i)}(\tilde{\epsilon}_i)$. I rewrite Eq. (5) as a summation,

$$\tilde{\epsilon}_M = \epsilon_M + \sum_{i=0}^{M-1} V(\epsilon_M, \epsilon_i) N(\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}), \quad (6)$$

where, based on the fact that the intervals $[\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}]$ are small and V is assumed to be continuous in both variables, I used the approximation $V(\epsilon_M, \epsilon_i) \approx V(\epsilon_M, \epsilon_{i-1})$ for any $i < M$. Using this decomposition I calculate the FES parameters.

First I calculate the *direct* exclusion statistics parameter by adding I_{M-1} particles in the interval $[\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M]$. Since I hold fix $\tilde{\epsilon}_{M-1}$ and $\tilde{\epsilon}_M$, then ϵ_{M-1} and all the energy levels below it will also remain fix, while ϵ_M and all the energy levels above it will

change. I calculate the change of single particle states in the interval $[\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M]$ (see figure 2, with $i = M - 1$ and $j = M$); I use the notation $\epsilon'_M \equiv \epsilon_{M, n(\tilde{\epsilon}' < \tilde{\epsilon}), I_{M-1}}(\tilde{\epsilon})$, which is the value taken by ϵ_M after the insertion of the I_{M-1} particles. The initial number of states in the interval $[\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M]$ is $G(\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M) = \int_{\epsilon_{M-1}}^{\epsilon_M} \sigma(\epsilon') d\epsilon'$ and after the addition of particles it changes into $G'(\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M) = \int_{\epsilon'_{M-1}}^{\epsilon'_M} \sigma(\epsilon') d\epsilon'$. So, to calculate the difference $\delta G(\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M) = G'(\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M) - G(\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M) = \int_{\epsilon_M}^{\epsilon'_M} \sigma(\epsilon') d\epsilon'$, I calculate first the change of ϵ'_M :

$$\tilde{\epsilon}_M = \epsilon'_M + V(\epsilon'_M, \epsilon_{M-1}) I_{M-1} + \sum_{i=0}^{M-1} V(\epsilon'_M, \epsilon_i) N(\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}) \quad (7)$$

If I denote $\delta\epsilon_M = \epsilon'_M - \epsilon_M$ and I expand $V(\epsilon_M, \epsilon_i)$ around ϵ_M , I get from (6) and (7) an equation for $\delta\epsilon$:

$$\delta\epsilon_M = \frac{-V(\epsilon_M, \epsilon_{M-1}) I_{M-1}}{1 + \frac{\partial V(\epsilon_M, \epsilon_{M-1})}{\partial \epsilon_M} I_{M-1} + \sum_{i=0}^{M-1} \frac{\partial V(\epsilon_M, \epsilon_i)}{\partial \epsilon_M} N(\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1})} \quad (8)$$

or, changing the summation into an integral,

$$\delta\epsilon = \frac{-V(\epsilon_M, \epsilon_{M-1}) I_{M-1}}{1 + \frac{\partial V(\epsilon, \epsilon)}{\partial \epsilon_M} I_{M-1} + \int_0^{\epsilon(\tilde{\epsilon})} \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon} \sigma(\epsilon') n(\epsilon') d\epsilon'}. \quad (9)$$

I look for linear effects, therefore I ignore the term proportional to I_{M-1} from the denominator of equation (9) and I replace $V(\epsilon_M, \epsilon_{M-1})$ by $V(\epsilon_M, \epsilon_M)$ (assuming that V is continuous in both variables). Writing $\delta G(\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M) = \delta\epsilon_M \cdot \sigma(\epsilon_M) \equiv \alpha_{\tilde{\epsilon}_M \tilde{\epsilon}_M} I_{M-1}$, I get the *direct* exclusion statistics parameter

$$\alpha_{\tilde{\epsilon}\tilde{\epsilon}} = \frac{V(\epsilon_M, \epsilon_M) \sigma[\epsilon(\tilde{\epsilon})]}{1 + \int_0^{\epsilon} \frac{\partial V[\epsilon, \epsilon']}{\partial \epsilon} \sigma(\epsilon') n(\epsilon') d\epsilon'}. \quad (10)$$

Note that $\alpha_{\tilde{\epsilon}\tilde{\epsilon}}$ is identical to α calculated before [17, 18, 14, 15, 16, 19] if $\partial V[\epsilon, \epsilon(\tilde{\epsilon})]/\partial \epsilon \equiv 0$.

Now let's calculate the *mutual* exclusion statistics parameters. For this I introduce I_i particles in the interval $[\tilde{\epsilon}_i, \tilde{\epsilon}_{i+1}]$ ($0 \leq i < M - 1$). This will change all the energy levels ϵ_j , of $j > i$ (see figure 2); let's denote the new values of ϵ_j , $j > i$, by ϵ'_j . Taking all these into account, I write

$$\tilde{\epsilon} = \epsilon' + V(\epsilon', \epsilon_i) I_i + \sum_{j=0}^i V[\epsilon', \epsilon_j] N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}) + \sum_{j=i+1}^{M-1} V[\epsilon', \epsilon'_j] N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}). \quad (11)$$

Expanding again $V(\epsilon, \epsilon')$ to the linear order in both variables, I get the equation for $\delta\epsilon_M^{(i)} \equiv (\epsilon'_M)^{(i)} - \epsilon_M$:

$$\delta\epsilon_M^{(i)} = - \frac{I_i V(\epsilon_M, \epsilon_i) + \sum_{j=i}^{M-1} \frac{\partial V(\epsilon_M, \epsilon_j)}{\partial \epsilon_j} N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}) \delta\epsilon_j}{1 + I \frac{\partial V(\epsilon_M, \epsilon_i)}{\partial \epsilon_M} + \sum_{j=0}^{M-1} \frac{\partial V(\epsilon_M, \epsilon_j)}{\partial \epsilon_M} N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1})}, \quad (12)$$

where I used the superscript to indicate that the particles were inserted at $\tilde{\epsilon}_i$. The unknown quantities, $\delta\epsilon_j^{(i)} = (\epsilon'_j)^{(i)} - \epsilon_j$, can be calculated recursively, starting from $j = i$, using first equation (9) and then equation (12). By doing so, we first notice that

$\delta\epsilon_j^{(i)}$ is proportional to I_i , for any j . Transforming both summations of equation (12) into integrals and introducing the notation

$$f(\tilde{\epsilon}_M, \tilde{\epsilon}_i) = \frac{\sum_{j=i}^{M-1} \frac{\partial V(\epsilon_M, \epsilon_j)}{\partial \epsilon_j} N(\tilde{\epsilon}_j, \tilde{\epsilon}_{j+1}) \delta\epsilon_j^{(i)}}{I_i} = \frac{\int_{\epsilon_i}^{\epsilon_M} \frac{\partial V(\epsilon_M, \epsilon')}{\partial \epsilon'} \sigma(\epsilon') n(\epsilon') (\delta\epsilon')^{(i)} d\epsilon'}{I_i}, \quad (13)$$

I get the final equation for $\delta\epsilon$,

$$\delta\epsilon(\tilde{\epsilon}_M, \tilde{\epsilon}_i) = -\frac{V(\epsilon_M, \epsilon_i) + f(\tilde{\epsilon}_M, \tilde{\epsilon}_i)}{1 + \int_0^{\epsilon_M} \frac{\partial V(\epsilon_M, \epsilon')}{\partial \epsilon_M} \sigma(\epsilon') n(\epsilon') d\epsilon'} I_i \quad (14)$$

If we plug in equation (13) into equation (14), the later becomes an integral equation for $\delta\epsilon(\tilde{\epsilon}, \tilde{\epsilon}_i)$.

Having now the expression for $\delta\epsilon_M^{(i)}$, we can calculate the change of the number of states in the interval $[\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M]$:

$$\delta G(\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M) = \sigma(\epsilon_M) \delta\epsilon_M - \sigma(\epsilon_{M-1}) \delta\epsilon_{M-1} \approx \left. \frac{d\sigma(\epsilon)}{d\epsilon} \right|_{\epsilon_M} (\epsilon_M - \epsilon_{M-1}) \delta\epsilon_M, \quad (15)$$

where we ignored $\delta\epsilon_M - \delta\epsilon_{M-1}$, since $\delta\epsilon_M$ is itself a small quantity. Notice that because both, ϵ_{M-1} and ϵ_M , vary at the insertion of particles at energies lower than ϵ_{M-1} , the variation of the number of quasiparticle states in the interval $[\tilde{\epsilon}_{M-1}, \tilde{\epsilon}_M]$ is proportional to $\epsilon_M - \epsilon_{M-1}$, i.e. is proportional to the dimension of the interval. Plugging equation (14) into equation (15) I obtain the *mutual* exclusion statistics parameter,

$$\alpha_{\tilde{\epsilon}_M \tilde{\epsilon}_i} = \frac{(\epsilon_M - \epsilon_{M-1}) \{V(\epsilon_M, \epsilon_i) + f(\tilde{\epsilon}, \tilde{\epsilon}_i)\}}{1 + \int_0^{\epsilon(\tilde{\epsilon})} \frac{\partial V(\epsilon, \epsilon')}{\partial \epsilon} \sigma(\epsilon') n(\epsilon') d\epsilon'} \left[\frac{d\sigma(\epsilon)}{d\epsilon} \right]_{\epsilon(\tilde{\epsilon})} \quad (16)$$

One can see immediately that if $d\sigma(\epsilon)/d\epsilon = 0$ for any ϵ , as it was in the case of constant density spectrum, $\alpha_{\tilde{\epsilon}_M \tilde{\epsilon}_i} = 0$ for any $\tilde{\epsilon}_M \neq \tilde{\epsilon}_i$ [17, 18, 14, 15, 16, 19].

Now we observe directly the surprising character of the mutual exclusion statistics, namely that it is proportional to the energy interval on which it acts, $(\epsilon_M - \epsilon_{M-1})$. In Ref. [26] I showed that this characteristics is necessary to ensure the self-consistency of the FES formalism, especially in the thermodynamic limit. The method to calculate the particle population for such exclusion statistics parameters is also given there.

3. Conclusions

Fractional exclusion statistics (FES) is usually considered as an “exotic” type of statistics, manifested in special types of systems. Contrary to this belief in this paper, by analysing a system with a very general model of interaction between the constituent particles, I showed that FES is rather the rule than the exception. FES is manifested in general in interacting systems. Moreover, I calculated the FES parameters of the model gas and I showed that the *mutual* exclusion statistics parameters are proportional to the subspace on which they act. This conclusion is also in contradiction with the usual definition of FES and therefore seems peculiar. But it is not so. In Ref. [26] I showed that the typical definition of the mutual exclusion parameters leads to inconsistencies in the thermodynamics calculations and, in order to eliminate these inconsistencies, the exclusion parameters must have exactly the properties deduced here.

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